MATH 4010 | Tutorial 2 | 22/01/2015

Eg 1: Show that ℓ^p $(1\le p<+\infty)$ is a Banach space with norm given by
$$\|x\| = \Big(\sum_{j=1}^{\infty}|\xi_j|^p\Big)^{\frac{1}{p}}.$$

Pf: ① $\|\cdot\|$ is a norm on ℓ^p.

It sufficies to prove the triangle ineq. since others are follows easily from the definition.

Claim: $\Big(\sum_{j=1}^{\infty}|\xi_j+\eta_j|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{j=1}^{\infty}|\xi_j|^p\Big)^{\frac{1}{p}} + \Big(\sum_{j=1}^{\infty}|\eta_j|^p\Big)^{\frac{1}{p}}$, $\forall\, x=\{\xi_j\}\in\ell^p$, $y=\{\eta_j\}\in\ell^p$

— Minkowski's ineq.

Pf of the claim:

Since $ab \le \frac{a^p}{p}+\frac{b^q}{q}$, $\forall\, a,b>0$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>0$.

(e^x is a convex fcn. $\Rightarrow ab = e^{\log a+\log b} = e^{\frac{1}{p}\log a^p+\frac{1}{q}\log b^q}$
$$\le \frac{1}{p}e^{\log a^p}+\frac{1}{q}e^{\log b^q} = \frac{a^p}{p}+\frac{b^q}{q}.\ )$$

Suppose $\sum_{j=1}^{\infty}|\tilde{\xi}_j|^p=1$ and $\sum_{j=1}^{\infty}|\tilde{\eta}_j|^q=1$,

then $\sum_{j=1}^{\infty}|\tilde{\xi}_j\tilde{\eta}_j| \le \sum_{j=1}^{\infty}\Big(\frac{|\tilde{\xi}_j|^p}{p}+\frac{|\tilde{\eta}_j|^q}{q}\Big) = \frac{1}{p}\sum_{j=1}^{\infty}|\tilde{\xi}_j|^p+\frac{1}{q}\sum_{j=1}^{\infty}|\tilde{\eta}_j|^q = \frac{1}{p}+\frac{1}{q}=1$

Let $\tilde{\xi}_j = \frac{\xi_j}{\big(\sum_{j=1}^{\infty}|\xi_j|^p\big)^{\frac{1}{p}}}$, $\tilde{\eta}_j = \frac{\eta_j}{\big(\sum_{j=1}^{\infty}|\eta_j|^q\big)^{\frac{1}{q}}}$. Then $\sum|\tilde{\xi}_j|^p = \sum|\tilde{\eta}_j|^q = 1$

Hence, $\sum_{j=1}^{\infty}|\tilde{\xi}_j\tilde{\eta}_j|\le 1$, i.e. $\sum_{j=1}^{\infty}\frac{|\xi_j\eta_j|}{\big(\sum_{j=1}^{\infty}|\xi_j|^p\big)^{\frac{1}{p}}\big(\sum_{j=1}^{\infty}|\eta_j|^q\big)^{\frac{1}{q}}}\le 1$

So far, we have proved the following Hölder inequality
$$\sum_{j=1}^{\infty}|\xi_j\eta_j| \le \Big(\sum_{j=1}^{\infty}|\xi_j|^p\Big)^{\frac{1}{p}}\Big(\sum_{j=1}^{\infty}|\eta_j|^q\Big)^{\frac{1}{q}},\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p,q>0.$$

Now, we prove the Minkowski ineq.

If $p=1$, then $\sum|\xi_j+\eta_j| \le \sum|\xi_j|+\sum|\eta_j|$

For $p>1$, set $\omega_j=\xi_j+\eta_j$, then $|\omega_j|^p = |\xi_j+\eta_j||\omega_j|^{p-1} \le |\xi_j||\omega_j|^{p-1}+|\eta_j||\omega_j|^{p-1}$

$\Rightarrow \sum|\omega_j|^p \le \sum|\xi_j||\omega_j|^{p-1}+\sum|\eta_j||\omega_j|^{p-1}$

By the Hölder's ineq. $\sum|\xi_j||\omega_j|^{p-1} \le \big(\sum|\xi_j|^p\big)^{\frac{1}{p}}\big(\sum|\omega_j|^{(p-1)q}\big)^{\frac{1}{q}}$ $\quad \frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p}$

So is $\sum|\eta_j||\omega_j|^{p-1} \le \big(\sum|\eta_j|^p\big)^{\frac{1}{p}}\big(\sum\quad\big) \le \big(\sum|\xi_j|^p\big)^{\frac{1}{p}}\big(\sum|\omega_j|^p\big)^{1-\frac{1}{p}}$

$\Rightarrow \cdots$

② Completeness

$d(x,y) = \|x-y\|$ is complete.

Let $\{x_n\}$ be a Cauchy seq. with $x_n = \{\xi_j^n\}$. in l^p.

Then $\forall \varepsilon > 0$, $\exists N(\varepsilon)$ s.t. $\forall n, m > N(\varepsilon)$,

$$\|x_n - x_m\| = \left(\sum_{j=1}^{\infty} |\xi_j^n - \xi_j^m|^p\right)^{\frac{1}{p}} < \varepsilon. \qquad (*)$$

So, for any fixed $j = 1, 2, \cdots$, $|\xi_j^n - \xi_j^m| < \varepsilon$, that is $\{\xi_j^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, by the completeness of $\mathbb{R}$, $\exists \xi_j \in \mathbb{R}$ s.t. $\xi_j^n \to \xi_j$ $n \to \infty$.

Define $x = \{\xi_j\}$. We claim $x \in l^p$ and $x_n \to x$ in l^p.

It follows from $(*)$ that, for any k,

$$\sum_{j=1}^{k} |\xi_j^n - \xi_j^m|^p < \varepsilon^p,$$

Let $m \to \infty$, then $\sum_{j=1}^{k} |\xi_j^n - \xi_j|^p \leq \varepsilon^p$

Let $k \to \infty$, then $\sum_{j=1}^{\infty} |\xi_j^n - \xi_j|^p \leq \varepsilon^p \Rightarrow x_n - x \in l^p \qquad (**)$

By Minkowski ineq. $x = x_n - (x_n - x) \in l^p$, $(**)$ show $x_n \to x$ as $n \to \infty$

Ex. 1: l^∞ is a Banach space with norm $\|x\| = \sup_j |\xi_j|$

Eg 2. Let X be a normed space. Then
X is complete iff $\sum_{n=1}^{\infty}\|x_n\| < +\infty \Rightarrow \sum_{n=1}^{\infty} x_n < +\infty$.

Remark: This example implies that for a complete normed space (Banach) absolute converge $\Rightarrow$ converge of series (in Mathematical anglysis Series!)

Pf: $\Rightarrow$ Suppose X is a complete normed space, then $\forall \{x_n\} \subset X$,
$$\left\|\sum_{n=m}^{m+p} x_n\right\| \leq \sum_{n=m}^{m+p}\|x_n\| \to 0 \text{ as } m \to +\infty.$$
i.e. $\|S_{m+p} - S_m\| \to 0$ as $m \to +\infty$ where $S_m = \sum_{n=1}^{m} x_n$.
$\{S_m\}$ is a Cauchy sequence in X. ~~By the~~ completeness of X
S_n converge in X. So $\sum_{n=1}^{\infty} x_n < +\infty$.

$\Leftarrow$ Let $\{x_n\}$ be a Cauchy sequence in X.

Claim: To prove $\{x_n\}$ is convergent, it sufficies to prove that $\exists \{x_{n_k}\}$ converge.

Pf of the claim: Let $\{x_{n_k} \to x$ in X.
Since $\{x_n\}$ is a Cauchy sequence, then $\forall \varepsilon > 0$, $\exists N(\varepsilon)$ s.t.
$$\|x_n - x_m\| < \varepsilon \quad \forall\, n, m \geq N(\varepsilon).$$
Since $n_k \to +\infty$ as $k \to \infty$, then $\exists K$ s.t. $n_k > N(\varepsilon)$ for $k \geq K$.
Then for $k \geq K$, and $n > N(\varepsilon)$
$$\|x_n - x_{n_k}\| < \varepsilon,$$
Let $k \to \infty$, then $\|x_n - x\| \leq \varepsilon$. So $x_n \to x$ in X.

By the claim, we only need to find a subsequence of $\{x_n\}$ s.t. $\{x_{n_k}\}$ is convergente.
$\forall k \in \mathbb{N}$, let $\varepsilon_k = \frac{1}{2^k}$, since $\{x_n\}$ is a Cauchy sequence, $\exists N_k$ s.t.
$\|x_n - x_m\| < \varepsilon_k = \frac{1}{2^k}$, $\forall n, m > N_k$, thus $\exists n_{k+1} \geq n_k > N_k$ s.t.
$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}$$
Set $y_k = x_{n_k}$, then $\sum_{i=1}^{\infty}\|y_{i+1} - y_i\| = \sum_{i=1}^{\infty}\|x_{n_{i+1}} - x_{n_i}\| < \sum_{i=1}^{\infty}\frac{1}{2^i} = 1$
$\Rightarrow y_k = y_1 + \sum_{i=1}^{k-1}(y_{i+1} - y_i)$ is convergent
Since $\sum_{i=1}^{\infty}\|y_{i+1} - y_i\| < +\infty \Rightarrow \sum_{i=1}^{\infty}(y_{i+1} - y_i) < +\infty$.
i.e. $\{x_{n_k}\}$ is converge.